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The critical exponents for the quasi-linear parabolic equations with inhomogeneous terms ☆

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Abstract

This paper deals with the critical exponents for the quasi-linear parabolic equations in $\mathbf{R}^n$ and with an inhomogeneous source, or in exterior domains and with inhomogeneous boundary conditions. For $n \geqslant 3$, $\sigma > -2/n$ and $p > \max\{1, 1+\sigma\}$, we obtain that $p_c = n(1+\sigma)/(n-2)$ is the critical exponent of these equations. Furthermore, we prove that if $\max\{1, 1+\sigma\} < p \leqslant p_c$, then every positive solution of these equations blows up in finite time; whereas these equations admit the global positive solutions for some $f(x)$ and some initial data $u_0(x)$ if $p > p_c$. Meantime, we also demonstrate that every positive solution of these equations blows up in finite time provided $n = 1, 2$, $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$.
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1. Introduction

The study of blow-up for nonlinear parabolic equations originates from Fujita [4]. He studies the Cauchy problem of the semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u = u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

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and obtains the following results:

(a) If $p < 1 + 2/n$, then every nontrivial solution of (1.1) blows up in finite time.
(b) If $p > 1 + 2/n$ and $u_0(x) \leqslant \delta e^{-|x|^2}$ $(0 < \delta \ll 1)$, then (1.1) admits a global solution.

In the case $p = 1 + 2/n$, it is shown by the author of [5] for $n = 1, 2$ and by the authors of [7] for $n \geqslant 1$ that the equation has no global solution satisfying $\|u(\cdot, t)\|_\infty < \infty$ for $t \geqslant 0$. The author of [16] proves that if $p = 1 + 2/n$, the equation has no global solution satisfying $\|u(\cdot, t)\|_q < \infty$ for $t > 0$ and some $q \in [1, +\infty)$. The value $p_c = 1 + 2/n$ is called the critical exponent of the semilinear heat equation (1.1). It plays an important role in the large-time behavior of the solutions to the semilinear heat equation (1.1).

In the past couple of years, there are a number of extensions of Fujita's results in many directions. See [1–3,8,12,14,19–23] and references therein for a detailed account of this aspect.

Recently, the authors of [2,20–23] extend Fujita's results to inhomogeneous equations. In this case, the value of the critical exponent and the location of blow-up points are not the same as those for the homogeneous equation. The critical exponent is more closely tied to the critical exponent of the corresponding elliptic problem. For example, the critical exponent $n/(n-2)$ is the same as the infimum of those p for which the problem $\Delta v + v^p = 0$ in $\mathbf{R}^n$ has singular solutions of the form $v = Ar^{-2/(p-1)}$. In particular, the authors of [2,21–23] make use of the Green's function of the heat equation to demonstrate that p_c belongs to the blow-up case.

For quasi-linear parabolic equations, the authors of [1,6,10–14, etc.] study the homogeneous equations and systems. For example, the author of [12] shows that $p_c = 1 + \sigma + 2/n$ is the critical exponent of the problem $u_t - \Delta u^{1+\sigma} = u^p$ in $\mathbf{R}^n$. Later, the authors of [11] show that p_c belongs to the blow-up case. The authors of [1,9] discuss the more general equations and the doubly singular parabolic equations, respectively, and obtain similar results.

In this paper, we investigate the critical exponents of the following problems:

$$\begin{cases} u_t - \Delta u^{1+\sigma} = u^p + f(x), & (x,t) \in \mathbf{R}^n \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

where $\sigma > -1$, $p > \max\{1, 1+\sigma\}$, and $f(x)$ and $u_0(x)$ are two nonnegative continuous functions in $\mathbf{R}^n$, and $f(x) \not\equiv 0$ in $\mathbf{R}^n$,

$$\begin{cases} u_t - \Delta u^{1+\sigma} = u^p, & (x,t) \in D^c \times (0, +\infty), \\ u(x,t) = f(x), & (x,t) \in \partial D \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in D^c, \end{cases} \tag{1.3}$$

$$\begin{cases} u_t - \Delta u^{1+\sigma} = u^p, & (x,t) \in D^c \times (0, +\infty), \\ \dfrac{\partial u^{1+\sigma}}{\partial \nu} = f(x), & (x,t) \in \partial D \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in D^c, \end{cases} \tag{1.4}$$

where $\sigma > -1$, $p > \max\{1, 1+\sigma\}$, D is a bounded domain in $\mathbf{R}^n$ with smooth boundary, and $D^c \triangleq \mathbf{R}^n \setminus \overline{D} = \mathbf{R}^n - \overline{D}$; ν is the outward unit normal to ∂D. $u_0(x)$ is a nonnegative continuous function in D^c, $f(x) \not\equiv 0$ is a nonnegative continuous function on ∂D, and $u_0(x) = f(x)$ on ∂D in problem (1.3) or $(\partial u_0^{1+\sigma})/\partial\nu = f(x)$ on ∂D in problem (1.4), i.e., $f(x)$ and $u_0(x)$ satisfy the compatibility condition. These equations of problems (1.2)–(1.4) arise from the nonlinear fluid dynamics (see [17]).

The problem (1.2) is partly investigated by the author of [21]. The author of [21] obtains that $p_c = n(1+\sigma)/(n-2)$ is the critical exponent of problem (1.2). However, the results of the author of [21] are silent about whether or not the critical exponent belongs to the blow-up case. In our paper, we will give a complete proof about the existence and nonexistence of global solutions for problem (1.2), and our method is different from that of [21].

The main interests of this paper are to study the large-time behavior of positive solutions of problems (1.2)–(1.4). We will show that $p_c = n(1+\sigma)/(n-2)$ is the critical exponent of these problems. It is easy to check that the critical exponent is not the same as that of [12], but equal to the infimum of those p for which the problem $-\Delta u^{1+\sigma} \geqslant u^p$ in $\mathbf{R}^n$ has solutions of the form $u = A(\chi + |x|)^{(1+\sigma)/[p-(1+\sigma)]}$ (see [15]).

Our methods are partly developed by the authors of [2,21–23]. However, the quasi-linear equations have no corresponding Green's function. We must look for other method. Fortunately, these quasi-linear equations have the Comparison Theorem. Thus, by making use of the method which lets the sub-solution 0 of the corresponding stationary problems become the initial data such that the positive solutions of the parabolic problems with the initial data 0 are monotone increasing to t, we will show that p_c belongs to the blow-up case.

Our results are as follows:

Theorem 1. *Suppose that $n \geqslant 3$, $\sigma > -\frac{2}{n}$ and $p > \max\{1, 1+\sigma\}$.*

(a) *If $p \leqslant \frac{n(1+\sigma)}{n-2}$, then every positive solution of* (1.2) *blows up in finite time.*
(b) *If $p > \frac{n(1+\sigma)}{n-2}$, then* (1.2) *admits a global positive solution for some $f(x)$ and $u_0(x)$.*

Theorem 2. *Suppose that $n \geqslant 3$, $\sigma > -\frac{2}{n}$ and $p > \max\{1, 1+\sigma\}$.*

(a) *If $p \leqslant \frac{n(1+\sigma)}{n-2}$, then every positive solution of* (1.3) *blows up in finite time.*
(b) *If $p > \frac{n(1+\sigma)}{n-2}$, then* (1.3) *admits a global positive solution for some $f(x)$ and $u_0(x)$.*

Theorem 3. *Suppose that $n \geqslant 3$, $\sigma > -\frac{2}{n}$ and $p > \max\{1, 1+\sigma\}$.*

(a) *If $p \leqslant \frac{n(1+\sigma)}{n-2}$, then every positive solution of* (1.4) *blows up in finite time.*
(b) *If $p > \frac{n(1+\sigma)}{n-2}$, then* (1.4) *admits a global positive solution for some $f(x)$ and $u_0(x)$.*

Remark. For $n = 1, 2$, $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$, we also show that every positive solution of problems (1.2)–(1.4) blows up in finite time. Its proof is included in Part (a) respectively.

This paper is organized as follows. In Section 2, we give preliminaries. In Sections 3, 4 and 5, we show Theorems 1, 2 and 3, respectively.

In the following, R and T are two given positive real numbers greater than one. C is a positive constant independent of R and T, and its value may change from line to line.

2. Preliminaries

In this section, we first give the definitions of solutions for problems (1.2), (1.3) and (1.4) respectively as those in [2,12,21], and then cite the Comparison Theorem and a known result. Lemma 2.1 is a special case of [15].

Definition 2.1.

(a) A continuous function $u = u(x,t)$ is called a solution of problem (1.2) in $Q_T \triangleq \mathbf{R}^n \times [0,T)$ if the following hold:
  (i) $\nabla_x u^{1+\sigma} \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$;
  (ii) for any bounded domain $D_1 \subset \mathbf{R}^n$ and for all $\psi \in C^2(D_1 \times [0,T))$ and vanishing on $\partial D_1 \times [0,T)$,

$$\int_0^\tau \int_{D_1} \left(u\partial_t\psi - \nabla u^{1+\sigma}\nabla\psi + u^p\psi + f\psi\right) dx\,dt - \int_{D_1} u(x,\cdot)\psi(x,\cdot)\big|_0^\tau\, dx = 0, \tag{2.1}$$

for all $\tau \in [0,T)$.

(b) A function $u = u(x,t)$ is a solution of problem (1.3) in $Q_T \triangleq D^c \times [0,T)$ if
  (i) $u \in C(0,T; L^1_{\mathrm{loc}}(D^c)) \cap C(0,T; L^p_{\mathrm{loc}}(D^c))$ and $\nabla_x u^{1+\sigma} \in L^2(0,T; L^2_{\mathrm{loc}}(D^c))$;
  (ii) for all compactly supported $\psi \in C^2(D^c \times [0,T)) \cap C^1(\overline{D^c} \times [0,T))$ and vanishing on $\partial D \times [0,T)$,

$$\begin{aligned}&\int_0^\tau \int_{D^c} \left(u\partial_t\psi + u^{1+\sigma}\Delta\psi + u^p\psi\right) dx\,dt - \int_0^\tau \int_{\partial D} f^{1+\sigma}\frac{\partial\psi}{\partial\nu}\, dS\,dt \\ &\quad - \int_{D^c} u(x,\cdot)\psi(x,\cdot)\big|_0^\tau\, dx = 0,\end{aligned} \tag{2.2}$$

for all $\tau \in [0,T)$.

(c) A function $u = u(x,t)$ is a solution of problem (1.4) in $Q_T \triangleq D^c \times [0,T)$ if
  (i) $u \in C(0,T; L^1_{\mathrm{loc}}(D^c)) \cap C(0,T; L^p_{\mathrm{loc}}(D^c))$ and $\nabla_x u^{1+\sigma} \in L^2(0,T; L^2_{\mathrm{loc}}(D^c))$;
  (ii) for all compactly supported $\psi \in C^2(D^c \times [0,T)) \cap C(\overline{D^c} \times [0,T))$,

$$\begin{aligned}&\int_0^\tau \int_{D^c} \left(u\partial_t\psi - \nabla u^{1+\sigma}\nabla\psi + u^p\psi\right) dx\,dt + \int_0^\tau \int_{\partial D} f\psi\, dS\,dt \\ &\quad - \int_{D^c} u(x,\cdot)\psi(x,\cdot)\big|_0^\tau\, dx = 0,\end{aligned} \tag{2.3}$$

for all $\tau \in [0,T)$.

Lemma 2.1. [15] *For any exterior domain $\mathbf{R}^n \setminus B_{R_0}(0)$, let u be a positive weak solution of the following problem*

$$-\Delta u \geqslant 0, \quad x \in \mathbf{R}^n \setminus B_{R_0}(0), \tag{2.4}$$

where R_0 is a given positive real number, and $B_{R_0}(0)$ is an open ball which centers at 0 and with the radius R_0. Then for $n \geqslant 3$, there exists a positive constant $C_0 = C_0(p,n,u|_{|x|=R_0},R_0)$ such that $u(x) \geqslant C_0|x|^{-(n-2)}$ in $\mathbf{R}^n \setminus B_{R_0}(0)$. Whereas for $n = 1,2$, there exists a positive constant c_0 such that $\lim_{|x|\to\infty}\inf u(x) = c_0 > 0$.

Lemma 2.2 *(The Comparison Theorem).* [17] *Let $u, v \in C(0,T; L^2_{\mathrm{loc}}(\Omega))$, $\nabla u^{1+\sigma}, \nabla v^{1+\sigma} \in L^2(0,T; L^2_{\mathrm{loc}}(\Omega))$, and satisfy*

$$\partial_t u - \Delta u^{1+\sigma} \leqslant \partial_t v - \Delta v^{1+\sigma}, \quad (x,t) \in \Omega_T, \tag{2.5}$$

$u \leqslant v$ in $\partial\Omega_T$, then $u \leqslant v$ for all $(x,t) \in \Omega_T$, where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ or $\Omega = \mathbf{R}^n$, and $\Omega_T = \Omega \times (0,T]$.

Remark 2.1. By a similar argument as Theorem 3.1 in [17], the Comparison Theorem also applies to the problems in the unbounded domains and with different boundary conditions.

Lemma 2.3 *(The monotonicity property). Let $\underline{u}(x)$ be a nonnegative sub-solution to the stationary problem of problem* (1.2). *Then the positive solution $u(x,t)$ of problem* (1.2) *with the initial data $\underline{u}(x)$ is monotone increasing to t.*

Proof. It is similar to that of Lemma 4.2.4 in [18]. From the assumption of Lemma 2.3, we know that $\underline{u}(x)$ is a sub-solution of problem (1.2) with the initial data $\underline{u}(x)$, thus by Lemma 2.2, we have $u(x,t) \geqslant \underline{u}(x)$ for all $t > 0$. Hence $u(x,\epsilon) \geqslant \underline{u}(x)$ for any $\epsilon > 0$. Let $u(x,\epsilon)$ and $\underline{u}(x)$ be the initial datum of problem (1.2) respectively, then by Lemma 2.2, the corresponding solutions of problem (1.2) satisfy that $u(x,t+\epsilon) \geqslant u(x,t)$, which implies that the positive solution to problem (1.2) with the initial data $\underline{u}(x)$ is monotone increasing to t. $\square$

Remark 2.2. Let $\underline{u}(x)$ be a nonnegative sub-solution to the stationary problem of (1.3) or (1.4). Then, by a similar argument, the positive solution $u(x,t)$ of problem (1.3) or (1.4) with the initial data $\underline{u}(x)$ is monotone increasing to t.

3. Proof of Theorem 1

In this section, we show Theorem 1. Our methods are partly different from that in [21]. In particular, by making use of the method which lets the sub-solution 0 of the corresponding stationary problems become the initial data such that the positive solutions of the parabolic problems with the initial data 0 are monotone increasing to t, we will show that $p \leqslant p_c$ belongs to the blow-up case.

Proof of Theorem 1. *Part* (a). We first consider the following problem

$$\begin{cases} \partial_t u - \Delta u^{1+\sigma} = u^p + f(x), & (x,t) \in \mathbf{R}^n \times (0,+\infty), \\ u(x,0) = 0, & x \in \mathbf{R}^n. \end{cases} \tag{3.1}$$

It is clear that the positive solution of problem (3.1) is a sub-solution of problem (1.2). If every positive solution of problem (3.1) blows up in finite time, then, by Lemma 2.2, every positive solution of problem (1.2) also blows up in finite time. Therefore, we only need consider problem (3.1).

The stationary problem of problem (3.1) is as follows

$$-\Delta u^{1+\sigma} = u^p + f(x), \quad x \in \mathbf{R}^n. \tag{3.2}$$

It is obvious that 0 is a sub-solution of problem (3.2) and does not satisfy problem (3.2). Thus, by making use of Lemmas 2.2 and 2.3, the positive solution of problem (3.1) is monotone increasing to t.

We argue by contradiction. Assume that problem (3.1) admits a global positive solution for $p \leqslant \frac{n(1+\sigma)}{n-2}$.

Let $\varphi(r)$ and $\eta(t)$ be two functions in $C^\infty([0,\infty))$, and satisfy:

(i) $0 \leqslant \varphi(r) \leqslant 1$ in $[0,\infty)$; $\varphi(r) \equiv 1$ in $[0,1]$, $\varphi(r) \equiv 0$ in $[2,\infty)$; $-C \leqslant \varphi'(r) \leqslant 0$, $|\varphi''(r)| \leqslant C$;
(ii) $0 \leqslant \eta(t) \leqslant 1$ in $[0,\infty)$; $\eta(t) \equiv 1$ in $[0,1]$; $\eta(t) \equiv 0$ in $[2,\infty)$; $-C \leqslant \eta'(t) \leqslant 0$.

For $R > 1$ and $T > 1$, define $Q_{R,T} = B_{2R}(0) \times [0, 4T]$, and let $\Psi(r,t) = \varphi_R(r)\eta_T(t)$ be a cut-off function, where $\varphi_R(r) = \varphi(\frac{r}{R})$, $\eta_T(t) = \eta(\frac{t}{2T})$. It is easy to check that

$$-\frac{C}{R} \leqslant \frac{\partial \varphi_R(r)}{\partial r} \leqslant 0, \qquad \left|\frac{\partial^2 \varphi_R(r)}{\partial r^2}\right| \leqslant \frac{C}{R^2}, \qquad -\frac{C}{T} \leqslant \frac{\partial \eta_T(t)}{\partial t} \leqslant 0. \tag{3.3}$$

Let $I_R = \int_{Q_{R,T}} u^p \Psi^s \,dx\,dt$, where $s > 1$ is a positive number to be determined. Then

$$\begin{aligned} I_R &= \int_{Q_{R,T}} u^p \Psi^s \,dx\,dt = \int_{Q_{R,T}} \left[-f(x) + u_t - \Delta u^{1+\sigma}\right]\Psi^s \,dx\,dt \\ &\leqslant -\int_{Q_{R,T}} f(x)\Psi^s \,dx\,dt - \int_{Q_{R,T}} u\varphi_R^s \frac{\partial \eta_T^s}{\partial t}\,dx\,dt - \int_{Q_{R,T}} u^{1+\sigma}\eta_T^s \Delta\varphi_R^s \,dx\,dt. \end{aligned} \tag{3.4}$$

By hypothesis of $f(x)$, we have $\int_{\mathbf{R}^n} f(x)\,dx > 0$. Thus, there exist $\delta > 0$ and $R_0 > 1$ such that $\int_{B_R(0)} f(x)\,dx \geqslant \delta$ for $R > R_0$. Hence, by the definition of φ_R and η_T, we have

$$I_R \leqslant -\delta T - \int_{2T}^{4T}\int_{B_{2R}(0)} u\varphi_R^s \frac{\partial \eta_T^s}{\partial t}\,dx\,dt - \int_0^{4T}\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\eta_T^s\Delta\varphi_R^s \,dx\,dt. \tag{3.5}$$

Since $\Delta\varphi_R^s = s\varphi_R^{s-1}\Delta\varphi_R + s(s-1)\varphi_R^{s-2}|\nabla\varphi_R|^2$ and

$$\Delta\varphi_R(r) = \frac{d^2\varphi_R(r)}{dr^2} + \frac{n-1}{r}\frac{d\varphi_R(r)}{dr}, \tag{3.6}$$

this, combined with (3.3), yields that $|\Delta\varphi_R^s| \leqslant \frac{C}{R^2}\varphi_R^{s-2}$ in $B_{2R}(0) \setminus \overline{B_R(0)}$.

Thus, (3.5) becomes

$$I_R \leqslant -\delta T + \frac{C}{T}\int_{2T}^{4T}\int_{B_{2R}(0)} u\Psi^{s-1}\,dx\,dt + \frac{C}{R^2}\int_0^{4T}\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\Psi^{s-2}\,dx\,dt. \tag{3.7}$$

Let s is large enough such that $p(s-1) \geqslant s$ and $\frac{(s-2)p}{1+\sigma} \geqslant s$. Then, by making use of the Young's inequality, we have

$$I_R \leqslant T\left(-\delta + CR^nT^{-\frac{p}{p-1}} + CR^{n-\frac{2p}{p-(1+\sigma)}}\right) + \frac{1}{2}I_R. \tag{3.8}$$

For $n \geqslant 3$, since $\sigma > -\frac{2}{n}$ and $\max\{1, 1+\sigma\} < p \leqslant \frac{n(1+\sigma)}{n-2}$, we have

$$n - \frac{2p}{p-(1+\sigma)} = \frac{(n-2)p - n(1+\sigma)}{p-(1+\sigma)} \leqslant 0.$$

For $n = 1, 2$, $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$, it is obvious that

$$n - \frac{2p}{p-(1+\sigma)} = \frac{(n-2)p - n(1+\sigma)}{p-(1+\sigma)} < 0.$$

Let $T \geqslant R^{n(p-1)/p}$ such that $R^n T^{-p/(p-1)} \leqslant 1$, then

$$I_R \leqslant CT, \quad \text{i.e.,} \quad \int_0^{4T} \int_{B_{2R}(0)} u^p \Psi^s \, dx \, dt \leqslant CT. \tag{3.9}$$

Thus

$$\int_T^{2T} \int_{B_R(0)} u^p(x,t) \, dx \, dt \leqslant CT. \tag{3.10}$$

By the integral mean value theorem, there exists $t_1 \in [T, 2T]$ such that

$$\int_{B_R(0)} u^p(x, t_1) \, dx \leqslant C, \tag{3.11}$$

where C is a positive constant independent of R and T. Since T is a large positive number and a random selection, and $u(x,t)$ is monotone increasing to t, there exists a positive number $T(R) > 1$ for any fixed $R > R_0$ such that for all $t > T(R)$,

$$\int_{B_R(0)} u^p(x,t) \, dx \leqslant C. \tag{3.12}$$

By the monotone increasing property of $u(x,t)$ to t, $\int_{B_R(0)} u^p(x,t)\,dx$ also is monotone increasing to t. This, combined with (3.12), yields that the limit $I_R^\infty \triangleq \lim_{t\to\infty} \int_{B_R(0)} u^p(x,t)\,dx$ exists, and

$$I_R^\infty \leqslant C. \tag{3.13}$$

Since $u(x,t)$ is nonnegative, I_R^∞ is monotone increasing to R. This, combined with (3.13), yields that $\lim_{R\to\infty} I_R^\infty$ exists. Thus, for any small $\varepsilon > 0$, there exists a large positive constant which still is denoted by R_0, such that for $R > R_0$,

$$\lim_{t\to\infty} \int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^p(x,t) \, dx \triangleq I_{2R}^\infty - I_R^\infty < \varepsilon. \tag{3.14}$$

Hence, by a similar argument as that in (3.12), there exists a large positive number $T(R) > 1$ such that

$$\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^p(x,t) \, dx < \varepsilon, \quad \text{for all } t > T(R). \tag{3.15}$$

On the other hand, we argue as that in [2,13]. Let $\phi(x) \in C^2(\mathbf{R}^n)$ be a positive function satisfying:

(i) $0 \leqslant \phi(x) \leqslant 1$ in $\mathbf{R}^n$; $\phi(x) \equiv 1$ in $B_1(0)$; $\phi(x) \equiv 0$ in $B_2^c(0)$;

(ii) $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial(B_2(0) \setminus B_1(0))$;
(iii) For any $\alpha \in (0,1)$, there exists a positive constant C_α such that $|\frac{\Delta\phi}{\phi^\alpha}| \leqslant C_\alpha$.

In fact, for any $\alpha \in (0,1)$, such a $\phi(x)$ exists. For example, let $\phi(x) = \exp(1 - \frac{1}{1-(1-|x|)^4})$ in $B_2(0) \setminus B_1(0)$, and extend $\phi(x)$ to $\mathbf{R}^n$ such that $\phi(x) \equiv 1$ in $B_1(0)$ and $\phi(x) \equiv 0$ in $B_2^c(0)$. It is easy to check that $\phi(x)$ satisfies the assumption above.

Let R and $T(R)$ be as defined in (3.14) and (3.15). Multiplying (3.1) by $\phi_R(x) = \phi(\frac{x}{R})$ and then integrating by parts in $\mathbf{R}^n$, we have

$$\frac{d}{dt}\int_{\mathbf{R}^n} u\phi_R(x)\,dx - \int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\Delta\phi_R(x)\,dx = \int_{\mathbf{R}^n} u^p\phi_R(x)\,dx + \int_{\mathbf{R}^n} f\phi_R(x)\,dx. \tag{3.16}$$

By the definition of $\phi_R(x)$, the Hölder's inequality and (3.15), we have

$$\left|\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\Delta\phi_R(x)\,dx\right| \leqslant C\left(\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^p\,dx\right)^{\frac{1+\sigma}{p}} R^{n-2-\frac{n(1+\sigma)}{p}} < C\varepsilon^{\frac{1+\sigma}{p}}. \tag{3.17}$$

Let $F_R(t) = \int_{\mathbf{R}^n} u\phi_R\,dx$ and $G(t) = \int_{\mathbf{R}^n} u^p\phi_R\,dx$. Then, by making use of (3.17) and $\int_{B_R(0)} f(x)\,dx \geqslant \delta$ for $R > R_0$, (3.16) becomes

$$F_R'(t) \geqslant G_R(t) - C\varepsilon^{\frac{1+\sigma}{p}} + \delta. \tag{3.18}$$

Thus, let ε be small enough such that $C\varepsilon^{(1+\sigma)/p} \leqslant \delta/2$, then $F_R'(t) \geqslant G_R(t) + \delta/2$.

Let $t_0 > T(R)$. By making use of the Hölder's inequality, we obtain that

$$\begin{aligned}\int_{t_0}^{t} \big(F_R(s)\big)^p\,ds &\leqslant C\int_{t_0}^{t} G_R(s)\,ds \leqslant CR^{\frac{np}{q}}\int_{t_0}^{t} F_R'(s)\,ds R^{\frac{np}{q}} \\ &= CR^{\frac{np}{q}}\big[F_R(t) - F_R(t_0)\big],\end{aligned} \tag{3.19}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $F_R(t) \geqslant 0$ for all $t \geqslant 0$, we have

$$F_R(t) \geqslant CR^{-\frac{np}{q}}\int_{t_0}^{t}\big(F_R(s)\big)^p\,ds + F(t_0) \geqslant CR^{-\frac{np}{q}}\int_{t_0}^{t}\big(F_R(s)\big)^p\,ds. \tag{3.20}$$

Let $g(t) = \int_{t_0}^{t}(F_R(s))^p\,ds$, then

$$g'(t) = \big(F_R(t)\big)^p \geqslant CR^{-\frac{np^2}{q}} g^p(t). \tag{3.21}$$

Let $t_1 > t_0$ such that $g(t_1) > 0$. Since $p > 1$, by solving the differential inequality (3.21) in $[t_1, t]$, we have

$$g(t) \geqslant \left(\frac{1}{C(-p+1)R^{-\frac{np^2}{q}}(t-t_1) + g^{-p+1}(t_1)}\right)^{\frac{1}{p-1}}. \tag{3.22}$$

Thus, there exists T_1 with $t_1 < T_1 \leqslant \frac{g^{-p+1}(t_1)}{C(p-1)R^{-np^2/q}}$, such that $\lim_{t\to T_1^-} g(t) = +\infty$, which implies that $g(t)$ and then u blow up in finite time. It contradicts our assumption. Therefore, every positive solution of problem (3.1) blows up in finite time. Hence, every positive solution of problem (1.2) blows up in finite time.

Part (b). In this part, we prove that for $p > \frac{n(1+\sigma)}{n-2}$, there exist some $f(x)$ and $u_0(x)$, such that problem (1.2) admits a global positive solution.

We first consider the stationary problem of problem (1.2) as follows

$$-\Delta u^{1+\sigma} = u^p + f(x), \quad x \in \mathbf{R}^n. \tag{3.23}$$

For $p > \frac{n(1+\sigma)}{n-2}$, by referring [15], there exists $v = C_1(\chi + |x|^2)^{-1/(p-(1+\sigma))}$ satisfying

$$-\Delta v^{1+\sigma} = v^p + C_2\big(\chi + |x|^2\big)^{-\frac{2p-(1+\sigma)}{p-(1+\sigma)}}, \quad x \in \mathbf{R}^n, \tag{3.24}$$

where χ, C_1 and C_2 are positive constants. Thus, if $f(x) \leqslant C_2(\chi + |x|^2)^{-(2p-(1+\sigma))/(p-(1+\sigma))}$ and $u_0(x) \leqslant C_1(\chi + |x|^2)^{-1/(p-(1+\sigma))}$, then v is a super-solution of problem (1.2). It is obvious that 0 is a sub-solution of problem (1.2). Therefore, by the iterative process and the Comparison Theorem, problem (1.2) admits a global positive solution. □

4. Proof of Theorem 2

In this section, by letting 0 be an initial data of the parabolic problems, and combining with the results of [15], we will demonstrate that every positive solution of problem (1.3) blows up in finite time for $p \leqslant \frac{n(1+\sigma)}{n-2}$.

Proof of Theorem 2. *Part* (a). We consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^{1+\sigma} = u^p, & (x,t) \in D^c \times (0,+\infty), \\ u(x,t) = f(x) f_1(t), & (x,t) \in \partial D \times (0,+\infty), \\ u(x,0) = 0, & x \in D^c, \end{cases} \tag{4.1}$$

where $f_1(t) \in C^1([0,\infty))$, $0 \leqslant f_1(t) \leqslant 1$ in $[0,\infty)$, $f_1(0) = 0$ and $f_1(t) = 1$ in $[1,\infty)$, such that $f(x) f_1(t)$ and $u(x,0)$ satisfy the compatibility condition.

It is obvious that positive solutions of problem (4.1) are sub-solutions of problem (1.3). If every positive solution of problem (4.1) blows up in finite time, then, by the Comparison Theorem, every positive solution of problem (1.3) also blows up in finite time. So we only need consider problem (4.1).

In the following, we first discuss the following elliptic problem

$$\begin{cases} -\Delta g = 0, & x \in D^c, \\ g = \big(f(x)\big)^{1+\sigma}, & x \in \partial D. \end{cases} \tag{4.2}$$

It is easy to check that $\max_{x\in\partial D}(f(x))^{1+\sigma} + 1$ and 0 are the super-solution and the sub-solution of problem (4.2), respectively. By the Comparison Theorem and the Maximum Principle for the elliptic equations, problem (4.2) admits a positive solution. For $n \geqslant 3$, by Lemma 2.1, there exist a large number $R_0 > 1$ and a positive constant C_0 such that $D \subset B_{R_0/2}(0)$, and $g(x) \geqslant C_0|x|^{-(n-2)}$ for all $|x| \geqslant R_0$. Hence, the following problem

$$\begin{cases} -\Delta h^{1+\sigma} = 0, & x \in D^c, \\ h = f(x), & x \in \partial D, \end{cases} \tag{4.3}$$

admits a positive solution, and $h(x) \geqslant (C_0)^{1/(1+\sigma)}|x|^{-(n-2)/(1+\sigma)}$ for all $|x| \geqslant R_0$.

It is clear that 0 is a sub-solution of problem (4.3) and does not satisfy problem (4.3). Thus, by making use of Remarks 2.1 and 2.2, the positive solution of the following problem is monotone increasing to t,

$$\begin{cases} \partial_t v - \Delta v^{1+\sigma} = 0, & x \in D^c, \\ v(x,t) = f(x) f_1(t), & (x,t) \in \partial D \times (0,+\infty), \\ v(x,0) = 0, & x \in D^c. \end{cases} \tag{4.4}$$

It is easy to check that $h(x)$ is a super-solution of problem (4.4). By the Comparison Theorem, we have $0 \leqslant v(x,t) \leqslant h(x)$. This, combined with the monotone increasing property of $v(x,t)$ to t, yields that $\lim_{t\to\infty} v(x,t)$ exists in D^c and $\lim_{t\to\infty} \partial_t v(x,t) = 0$. Furthermore, by the relation of problems (4.3) and (4.4), we have $\lim_{t\to\infty} v(x,t) = h(x)$ in D^c. Therefore, for any $R > R_0$, there exists a positive number $T(R,R_0) > 0$ such that for $x \in \overline{B_R(0) \setminus B_{R_0}(0)}$ and $t > T(R,R_0)$,

$$v(x,t) \geqslant \frac{1}{2} C_0^{\frac{1}{1+\sigma}} |x|^{-\frac{n-2}{1+\sigma}}. \tag{4.5}$$

We know that the positive solution of problem (4.4) is a sub-solution of problem (4.1). Thus, the positive solution of problem (4.1) satisfies

$$u(x,t) \geqslant \frac{1}{2} C_0^{\frac{1}{1+\sigma}} |x|^{-(n-2)/(1+\sigma)} \tag{4.6}$$

for $x \in \overline{B_R(0) \setminus B_{R_0}(0)}$ and $t > T(R,R_0)$.

For $n = 1, 2$, by similar arguments as those in (4.2)–(4.6) and by making use of Lemma 2.1, the positive solution of problem (4.1) satisfies

$$u(x,t) \geqslant \frac{1}{2} c_0^{\frac{1}{1+\sigma}}, \tag{4.7}$$

for $x \in \overline{B_R(0) \setminus B_{R_0}(0)}$ and $t > T(R,R_0)$.

Next, we argue by a contradiction. Assume that problem (4.1) has a global positive solution for $p \leqslant \frac{n(1+\sigma)}{n-2}$.

Let φ and η be as defined in Part (a) of Section 3, $R_0 > 1$ be a fixed large number such that $D \subset B_{R_0/2}(0)$, and $\psi(r) \in C^\infty([0,\infty))$ be a function satisfying: $0 \leqslant \psi(r) \leqslant 1$ in $[0,\infty)$; $\psi(r) = 0$ on $[0,1/2]$, $\psi(r) = 1$ in $[1,\infty)$; $0 \leqslant \psi'(r) \leqslant C$ and $|\psi''(r)| \leqslant C$.

For $R > R_0$, define $Q_{R,T} = (B_{2R}(0) \setminus \overline{D}) \times [0,4T]$, and let $\Psi = \varphi_R(r)\psi_{R_0}(r)\eta_T(t)$, where $\phi_R(r) = \phi(\frac{r}{R})$, $\psi_{R_0}(x) = \psi(\frac{r}{R_0})$ and $\eta_T(t) = \eta(\frac{t}{2T})$. Clearly,

$$-\frac{C}{R} \leqslant \frac{\partial \varphi_R(r)}{\partial r} \leqslant 0, \qquad \left|\frac{\partial^2 \varphi_R(r)}{\partial r^2}\right| \leqslant \frac{C}{R^2}, \qquad -\frac{C}{T} \leqslant \frac{\partial \eta_T(t)}{\partial t} \leqslant 0, \tag{4.8}$$

$$0 \leqslant \frac{\partial \psi_{R_0}(r)}{\partial r} \leqslant \frac{C}{R_0}, \qquad \left|\frac{\partial^2 \psi_{R_0}(r)}{\partial r^2}\right| \leqslant \frac{C}{R_0^2}. \tag{4.9}$$

$\Psi = 0$ for $x \in \overline{B_{R_0/2}(0)) \setminus D}$ or $x \in B_{2R}^c(0)$ or $t \in [4T,\infty)$, $\Psi = \psi_{R_0}(x)\eta_T(t)$ for $x \in B_{R_0}(0) \setminus \overline{D}$ and $\Psi = \varphi_R(r)\eta_T(t)$ for $x \in B_{2R}(0) \setminus \overline{B_{R_0}(0)}$.

Multiplying (4.1) by Ψ^s and then integrating by parts in $D^c \times [0,\infty)$, where $s > 1$ is a positive number to be determined, we have

$$\int\limits_{Q_{R,T}} u^p \Psi^s \, dx\, dt = \int\limits_{Q_{R,T}} \left[u_t - \Delta u^{1+\sigma}\right] \Psi^s \, dx\, dt$$

$$= \int_{B_{2R}(0)\setminus\overline{B_{R_0/2}(0)}} u\Psi^s\big|_0^{4T}\,dx - \int_{Q_{R,T}} u\varphi_R^s\psi^s(x)\frac{\partial\eta_T^s}{\partial t}\,dx\,dt - \int_{Q_{R,T}} u^{1+\sigma}\Delta\Psi^s\,dx\,dt. \tag{4.10}$$

By the definition of Ψ, and by making use of (4.7) and (4.8), we have

$$\int_{Q_{R,T}} u^p\Psi^s\,dx\,dt \leqslant \frac{C}{T}\int_{2T}^{4T}\int_{B_{2R}(0)\setminus\overline{B_{R_0/2}(0)}} u\Psi^{s-1}\,dx\,dt + \int_0^{4T}\int_{B_{R_0}(0)\setminus\overline{B_{R_0/2}(0)}} u^{1+\sigma}\eta_T^s\big|\Delta\psi_{R_0}^s\big|\,dx\,dt + \int_0^{4T}\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\eta_T^s\big|\Delta\varphi_R^s\big|\,dx\,dt. \tag{4.11}$$

Since $\Delta\varphi_R^s = s\varphi_R^{s-1}\Delta\varphi_R + s(s-1)\phi_R^{s-2}|\nabla\varphi_R|^2$ and $\Delta\varphi_R(r) = \frac{d^2\varphi_R(r)}{dr^2} + \frac{n-1}{r}\frac{d\varphi_R(r)}{dr}$, we have $|\Delta\varphi_R^s| \leqslant \frac{C}{R^2}\varphi_R^{s-2}$ in $B_{2R}(0)\setminus\overline{B_R(0)}$. By a similar argument, we have $|\Delta\psi_{R_0}^s| \leqslant \frac{C}{R_0^2}\psi_{R_0}^{s-2}$ in $B_{R_0}(0)\setminus\overline{B_{R_0/2}(0)}$. These, combined with (4.8), (4.9) and (4.11), yield that

$$\int_{Q_{R,T}} u^p\Psi^s\,dx\,dt \leqslant \frac{C}{T}\int_{2T}^{4T}\int_{B_{2R}(0)\setminus\overline{B_{R_0/2}(0)}} u\Psi^{s-1}\,dx\,dt + \frac{C}{R_0^2}\int_0^{4T}\int_{B_{R_0}(0)\setminus\overline{B_{R_0/2}(0)}} u^{1+\sigma}\Psi^{s-2}\,dx\,dt + \frac{C}{R^2}\int_0^{4T}\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\Psi^{s-2}\,dx\,dt. \tag{4.12}$$

Let s be large enough such that $(s-1)p > s$ and $\frac{(s-2)p}{1+\sigma} > s$. Then, by making use of the Young's inequality to (4.12), we have

$$\int_{Q_{R,T}} u^p\Psi^s\,dx\,dt \leqslant \frac{1}{2}\int_{Q_{R,T}} u^p\Psi^s\,dx\,dt + CT\big(R^nT^{-\frac{p}{p-1}} + R_0^{n-\frac{2p}{p-(1+\sigma)}} + R^{n-\frac{2p}{p-(1+\sigma)}}\big). \tag{4.13}$$

Thus

$$\int_{Q_{R,T}} u^p\Psi^s\,dx\,dt \leqslant CT\big(R^nT^{-\frac{p}{p-1}} + R_0^{n-\frac{2p}{p-(1+\sigma)}} + R^{n-\frac{2p}{p-(1+\sigma)}}\big). \tag{4.14}$$

For $n \geqslant 3$, since $\sigma > -\frac{2}{n}$ and $\max\{1, 1+\sigma\} < p \leqslant \frac{n(1+\sigma)}{n-2}$, we have

$$n - \frac{2p}{p-(1+\sigma)} = \frac{(n-2)p - n(1+\sigma)}{p-(1+\sigma)} \leqslant 0.$$

For $n = 1, 2$, $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$, it is obvious that

$$n - \frac{2p}{p-(1+\sigma)} = \frac{(n-2)p - n(1+\sigma)}{p-(1+\sigma)} < 0.$$

Let $T \geqslant R^{n(p-1)/p}$ such that $R^n T^{-p/(p-1)} \leqslant 1$, then (4.14) becomes

$$\int_{Q_{R,T}} u^p \Psi^s \, dx\, dt \leqslant CT, \tag{4.15}$$

where C is a positive constant independent of R, R_0 and T. Hence, we have

$$\int_T^{2T} \int_{B_R(0)\setminus\overline{B_{R_0}(0)}} u^p \, dx\, dt \leqslant CT. \tag{4.16}$$

By the integral mean value theorem, there exists $t_1 \in [T, 2T]$ such that

$$\int_{B_R(0)\setminus\overline{B_{R_0}(0)}} u^p(x, t_1)\, dx \leqslant C. \tag{4.17}$$

We know that T is a large number and a random selection, and $u(x,t)$ is monotone increasing to t. Thus there exists a large positive number $T(R, R_0)$ such that for $t > T(R, R_0)$,

$$\int_{B_R(0)\setminus B_{R_0}(0)} u^p(x, t)\, dx\, dt \leqslant C. \tag{4.18}$$

On the other hand, for $n \geqslant 3$, by making use of (4.6), we have

$$\begin{aligned}
\int_{B_R(0)\setminus B_{R_0}(0)} u^p(x,t)\, dx &\geqslant \omega_n \left(\frac{1}{2} C_0^{\frac{1}{1+\sigma}}\right)^p \int_{R_0}^{R} r^{-\frac{(n-2)p}{1+\sigma}} r^{n-1}\, dr \\
&\geqslant \omega_n \min\bigl\{1, 2^{-p_c} C_0^{\frac{p_c}{1+\sigma}}\bigr\} \int_{R_0}^{R} r^{-1}\, dr \\
&= \omega_n \min\bigl\{1, 2^{-p_c} C_0^{\frac{p_c}{1+\sigma}}\bigr\} \ln \frac{R}{R_0}
\end{aligned} \tag{4.19}$$

for $t > T(R, R_0)$, where ω_n is the area of the unit sphere with the dimension n.

Let R be large enough such that $\omega_n \min\{1, 2^{-p_c} C_0^{p_c/(1+\sigma)}\} \ln \frac{R}{R_0} \geqslant 2C$, then (4.19) satisfies

$$\int_{B_R(0)\setminus B_{R_0}(0)} u^p(x, t)\, dx \geqslant 2C \tag{4.20}$$

for $t > T(R, R_0)$, which contradicts (4.18).

For $n = 1, 2$, by a similar argument as above and by making use of (4.7), we have

$$\int\limits_{B_R(0)\setminus B_{R_0}(0)} u^p(x,t)\,dx \geqslant \omega_n 2^{-p} c_0^{\frac{p}{1+\sigma}} \frac{R^n - R_0^n}{n}. \tag{4.21}$$

Let R be large enough such that $\omega_n 2^{-p} c_0^{p/(1+\sigma)} \frac{R^n - R_0^n}{n} > 2C$, then

$$\int\limits_{B_R(0)\setminus B_{R_0}(0)} u^p(x,t)\,dx \geqslant 2C \tag{4.22}$$

for $t > T(R, R_0)$, which contradicts (4.18).

In conclusion, for $n \geqslant 3$, $\sigma > -\frac{2}{n}$ and $\max\{1, 1+\sigma\} < p \leqslant \frac{n(1+\sigma)}{n-2}$, or for $n = 1, 2$, $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$, every positive solution of problem (4.1) blows up in finite time. Hence every positive solution of problem (1.3) also blows up in finite time.

Part (b). By a similar argument as that in Part (b) of Section 3, let $v(x)$ be as defined in Part (b) of Section 3. If $f(x) \leqslant v(x)$ on ∂D and $u_0(x) \leqslant v(x)$ in D^c, then $v(x)$ and 0 are the super-solution and sub-solution of problem (1.3), respectively. Therefore, by the iterative process and the Comparison Theorem, problem (1.3) admits a global positive solution. □

5. Proof of Theorem 3

In this section, by a similar argument as that in Section 3, we show Theorem 3.

Proof of Theorem 3. *Part* (a). We consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^{1+\sigma} = u^p, & (x,t) \in D^c \times (0, +\infty), \\ \dfrac{\partial u^{1+\sigma}(x,t)}{\partial \nu} = f(x) f_1(t), & (x,t) \in \partial D \times (0, +\infty), \\ u(x,0) = 0, & x \in D^c, \end{cases} \tag{5.1}$$

where $f_1(t)$ is as defined in Part (a) of Section 4.

It is obvious that positive solutions of problem (5.1) are sub-solutions of problem (1.4). If every positive solution of problem (5.1) blows up in finite time, then by the Comparison Theorem, every positive solution of problem (1.4) also blows up in finite time. Therefore, we only need consider problem (5.1).

The stationary problem of problem (5.1) is as follows

$$\begin{cases} -\Delta u^{1+\sigma} = u^p, & x \in D^c, \\ \dfrac{\partial u^{1+\sigma}}{\partial \nu} = f(x), & x \in \partial D. \end{cases} \tag{5.2}$$

It is obvious that 0 is a sub-solution of problem (5.2) and does not satisfy problem (5.2). Thus, by making use of Remarks 2.1 and 2.2, the positive solution of problem (5.1) is monotone increasing to t.

We argue by a contradiction. Assume that problem (5.1) has a global positive solution for $p \leqslant \frac{n(1+\sigma)}{n-2}$.

Let $\varphi(r)$ and $\eta(t)$ be as defined in Part (a) of Section 3, and $R > 1$ and $T > 1$ be two large numbers such that $D \subset B_R(0)$. Define $Q_{R,T} \triangleq (B_{2R}(0) \setminus \overline{D}) \times [0, 4T]$ and let $\Psi(r,t) = \varphi_R(r)\eta_T(t)$ be a cut-off function, where $\varphi_R(r) = \varphi(\frac{r}{R})$, $\eta_T(t) = \eta(\frac{t}{2T})$. Clearly,

$$-\frac{C}{R} \leqslant \frac{\partial \varphi_R(r)}{\partial r} \leqslant 0, \qquad \left|\frac{\partial^2 \varphi_R(r)}{\partial r^2}\right| \leqslant \frac{C}{R^2}, \qquad -\frac{C}{T} \leqslant \frac{\partial \eta_T(t)}{\partial t} \leqslant 0. \tag{5.3}$$

Let $I_R = \int_{Q_{R,T}} u^p \Psi^s \, dx \, dt$, where $s > 1$ is a positive number to be determined. Then

$$\begin{aligned} I_R &= \int\limits_{Q_{R,T}} u^p \Psi^s \, dx \, dt = \int\limits_{Q_{R,T}} \left(u_t - \Delta u^{1+\sigma}\right) \Psi^s \, dx \, dt \\ &\leqslant -\int\limits_0^{4T} \int\limits_{\partial D} f(x) f_1(t) \Psi^s \, dS \, dt - \int\limits_{Q_{R,T}} u \varphi_R^s \frac{\partial \eta_T^s}{\partial t} \, dx \, dt - \int\limits_{Q_{R,T}} u^{1+\sigma} \eta_T^s \Delta \varphi_R^s \, dx \, dt. \end{aligned} \tag{5.4}$$

Since $f(x) \geqslant, \not\equiv 0$ on ∂D, there exists a positive number δ such that $\delta = \int_{\partial D} f(x) \, dS$. Thus, by the definition of $f_1(t)$, φ_R and η_T, we have

$$I_R \leqslant -\delta T - \int\limits_{2T}^{4T} \int\limits_{B_{2R}(0) \setminus \overline{D}} u \varphi_R^s \frac{\partial \eta_T^s}{\partial t} \, dx \, dt - \int\limits_0^{4T} \int\limits_{B_{2R}(0) \setminus \overline{B_R(0)}} u^{1+\sigma} \eta_T^s \Delta \varphi_R^s \, dx \, dt. \tag{5.5}$$

By a similar argument as (3.6), we have $|\Delta \phi_R^s| \leqslant \frac{C}{R^2} \phi_R^{s-2}$ in $B_{2R}(0) \setminus \overline{B_R(0)}$.

Thus, (5.5) becomes

$$I_R \leqslant -\delta T + \frac{C}{T} \int\limits_{2T}^{4T} \int\limits_{B_{2R}(0) \setminus \overline{D}} u \Psi^{s-1} \, dx \, dt + \frac{C}{R^2} \int\limits_0^{4T} \int\limits_{B_{2R}(0) \setminus \overline{B_R(0)}} u^{1+\sigma} \Psi^{s-2} \, dx \, dt. \tag{5.6}$$

Let s be large enough such that $p(s-1) \geqslant s$ and $\frac{(s-2)p}{1+\sigma} \geqslant s$. Then, by making use of the Young's inequality, we have

$$I_R \leqslant T\left(-\delta + C R^{n - \frac{2p}{p-(1+\sigma)}} + C R^n T^{-\frac{p}{p-1}}\right) + \frac{1}{2} I_R. \tag{5.7}$$

For $n \geqslant 3$, since $\sigma > -\frac{2}{n}$ and $\max\{1, 1+\sigma\} < p \leqslant \frac{n(1+\sigma)}{n-2}$, we have

$$n - \frac{2p}{p-(1+\sigma)} = \frac{(n-2)p - n(1+\sigma)}{p-(1+\sigma)} \leqslant 0.$$

For $n = 1, 2$, $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$, it is obvious that

$$n - \frac{2p}{p-(1+\sigma)} = \frac{(n-2)p - n(1+\sigma)}{p-(1+\sigma)} < 0.$$

Let $T \geqslant R^{n(p-1)/p}$ such that $R^n T^{-p/(p-1)} \leqslant 1$, then (5.7) becomes

$$I_R \leqslant CT, \quad \text{i.e.,} \quad \int\limits_0^{4T} \int\limits_{B_{2R}(0) \setminus \overline{D}} u^p \Psi_R^s \, dx \, dt \leqslant CT. \tag{5.8}$$

Thus, we have

$$\int_T^{2T} \int_{B_R(0)\setminus\overline{D}} u^p(x,t)\,dx\,dt \leqslant CT. \tag{5.9}$$

By the integral mean value theorem, there exists $t_1 \in [T, 2T]$ such that

$$\int_{B_R(0)\setminus\overline{D}} u^p(x,t_1)\,dx \leqslant C, \tag{5.10}$$

where C is independent of R and T. We know that T is a large number and a random selection, and $u(x,t)$ is monotone increasing to t. Thus, there exists a large positive number $T(R)$ such that for $t > T(R)$,

$$\int_{B_R(0)\setminus\overline{D}} u^p(x,t)\,dx \leqslant C. \tag{5.11}$$

Since $u(x,t)$ is monotone increasing to t, $\int_{B_R(0)\setminus\overline{D}} u^p(x,t)\,dx$ is also monotone increasing to t. This, combined with (5.11), yields that $I_R^\infty = \lim_{t\to\infty} \int_{B_R(0)\setminus\overline{D}} u^p(x,t)\,dx$ exists, and

$$I_R^\infty \leqslant C. \tag{5.12}$$

Since $u(x,t)$ is nonnegative, I_R^∞ is monotone increasing to R. This, combined with (5.12), yields that the limit $\lim_{R\to\infty} I_R^\infty$ exists. Thus, for any small $\varepsilon > 0$, there exists $R_0 > 1$ such that $D \subset B_{R_0}(0)$, and for $R > R_0$,

$$\lim_{t\to\infty} \int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^p(x,t)\,dx \triangleq I_{2R}^\infty - I_R^\infty < \varepsilon. \tag{5.13}$$

By a similar argument as that in (5.11), there exists a large positive number $T(R)$ such that

$$\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^p(x,t)\,dx < \varepsilon, \quad \text{for all } t > T(R). \tag{5.14}$$

On the other hand, let $\phi \in C^2(\mathbf{R}^n)$ be as defined in Part (a) of Section 3 and $\phi_R(x) = \phi(\frac{x}{R})$. Multiplying (5.1) by $\phi_R(x)$ and then integrating by parts in D^c, we have

$$\frac{d}{dt}\int_{D^c} u\phi_R(x)\,dx - \int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\Delta\phi_R(x)\,dx = \int_{D^c} u^p\phi_R(x)\,dx + \delta \tag{5.15}$$

for $t > T(R)$, where $\delta = \int_{\partial D} f(x)\,dS$. By making use of the Hölder's inequality, combined with (5.14), and noting that $n - 2 - \frac{n(1+\sigma)}{p} \leqslant 0$, we have

$$\left| \int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^{1+\sigma}\Delta\phi_R(x)\,dx \right| \leqslant C\Bigg(\int_{B_{2R}(0)\setminus\overline{B_R(0)}} u^p\,dx \Bigg)^{\frac{1+\sigma}{p}} R^{n-2-\frac{n(1+\sigma)}{p}} < C\varepsilon^{\frac{1+\sigma}{p}}. \tag{5.16}$$

Let $F_R(t)=\int_{D^c} u\phi_R(x)\,dx$ and $G_R(t)=\int_{D^c} u^p\phi_R(x)\,dx$. By (5.15) and (5.16), we have

$$F_R'(t)\geqslant G_R(t)-C\varepsilon^{\frac{1+\sigma}{p}}+\delta. \tag{5.17}$$

Let ε be sufficiently small such that $C\varepsilon^{(1+\sigma)/p}\leqslant\delta/2$, then $F_R'(t)\geqslant G_R(t)+\delta/2$. Thus, by similar arguments as those in (3.19)–(3.22), we obtain that every positive solution of (5.1) blows up in finite time. Hence, every positive solution of (1.4) blows up in finite time.

Part (b). For simplicity, we assume that $D=B_{R_0}(0)$, where R_0 be a positive constant.

Let $R>R_0$ and $u_1(x)=\varepsilon(\chi+|r-R|^2)^{-1/(p_1-1)}$, where ε and χ are positive constants to be determined, $p_1=\frac{p}{1+\sigma}$, $R_0<R<\frac{nR_0}{n-1}$ and $r=|x|$. It is easy to check that

$$\frac{\partial u_1}{\partial\nu}=\frac{\partial u_1}{\partial r}=\left(\frac{2\varepsilon}{p_1-1}\right)\bigl(\chi+(R-R_0)^2\bigr)^{-\frac{p_1}{p_1-1}}(R-R_0)>0,\quad \text{on } \partial D, \tag{5.18}$$

$$\begin{aligned}
-\Delta u_1&=-\left(\frac{\partial^2 u_1}{\partial r^2}+\frac{n-1}{r}\frac{\partial u_1}{\partial r}\right)\\
&=\left(\frac{2\varepsilon}{p_1-1}\right)\left[n-\frac{2p_1}{p_1-1}+\frac{2p_1\chi}{(p_1-1)(\chi+|r-R|^2)}-\frac{(n-1)R}{r}\right]\\
&\quad\times\bigl(\chi+|r-R|^2\bigr)^{-\frac{p_1}{p_1-1}},
\end{aligned} \tag{5.19}$$

in D^c. Since $p_1=\frac{p}{1+\sigma}>\frac{n}{n-2}$, we have $n-\frac{2p_1}{p_1-1}>0$. Let χ be sufficiently large and $r_0\geqslant\frac{2(n-1)R}{n-2p_1/(p_1-1)}$. Then, there exists a positive constant δ such that for $R_0\leqslant r\leqslant r_0$,

$$n-\frac{2p_1}{p_1-1}+\frac{2p_1\chi}{(p_1-1)(\chi+|r-R|^2)}-\frac{(n-1)R}{r}\geqslant n-\varepsilon_1-\frac{(n-1)R}{r}\geqslant\delta,$$

where ε_1 is a sufficient small positive number, whereas for $r>r_0$,

$$n-\frac{2p_1}{p_1-1}+\frac{2p_1\chi}{(p_1-1)(\chi+|r-R|^2)}-\frac{(n-1)R}{r}\geqslant n-\frac{2p_1}{p_1-1}-\frac{(n-1)R}{r}\geqslant\delta.$$

Thus,

$$-\Delta u_1\geqslant\left(\frac{2\varepsilon}{p_1-1}\right)\delta\bigl(\chi+|R-r|^2\bigr)^{-\frac{p_1}{p_1-1}}. \tag{5.20}$$

Since $p_1>1$, let ε satisfy that $(\frac{2\varepsilon}{p_1-1})\delta\geqslant\varepsilon^{p_1}$. Thus $-\Delta u_1\geqslant u_1^{p_1}$ in D^c.

Hence, setting $\tilde{u}^{1+\sigma}(x)=u_1$, we have

$$-\Delta\tilde{u}^{1+\sigma}\geqslant\tilde{u}^p,\quad \text{in } D^c. \tag{5.21}$$

Further, if $f(x)\leqslant(\frac{2\varepsilon}{p_1-1})(\chi+(R-R_0)^2)^{-p_1/(p_1-1)}(R-R_0)$ on ∂D and $u_0(x)\leqslant[\varepsilon(\chi+|r-R|^2)^{-1/(p_1-1)}]^{1/(1+\sigma)}$ in D^c, then $\tilde{u}(x)$ and 0 are the super-solution and sub-solution of problem (1.4), respectively. Therefore, by the iterative process and the Comparison Theorem, problem (1.4) admits a global solution for $p>\frac{n(1+\sigma)}{n-2}$. □

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